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Axially-symmetric eddies embedded in a rotational stream

By S. LEIBOVICH
Department of Thermal Engineering, Cornell University, Ithaca, New York

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The structure of eddies embedded in a swirling stream of an incompressible, inviscid fluid is examined. Laminar motion is assumed, as is axial symmetry of the flow field.

It is supposed that the eddies are formed either (i) as wakes behind solid obstacles placed in the stream, or (ii) as the products of vortex breakdown occurrences, in which case the eddies are completely surrounded by fluid of the outer stream. In either event, it is assumed that the presence of an eddy only slightly disturbs the motion of the outer stream, i.e. that the eddies are slender. In particular, this requires that $r_0 \Omega_\infty / W_\infty \ll 1$, where r_0 is the maximum radial extent of the eddy, and Ω_∞ and W_∞ are defined below.

The analysis is carried to completion when the undisturbed outer stream is a wheel flow. Principal results derived are summarized below. (1) Eddy lengths are related directly and simply to the internal vorticity, providing that there is swirl in the outer flow. (2) A corollary is that very slender eddies have a maximum possible length. In the case (ii) above, this length is $2\pi W_\infty/\Omega_\infty$, where W_∞ is the axial velocity and Ω_∞ the angular velocity of the undisturbed outer flow. (3) The shapes of the free streamlines bounding the eddy are calculated. A feature is the cusped nature of an eddy free end. (4) The motion inside the eddy and the disturbance to the outer stream are calculated.

Irrotational base flows are exceptional. Results (1) and (2) above cannot be found for a potential outer flow, but (3) and (4) remain.

1. Introduction

This paper deals with the steady axisymmetric motions around and within finite eddies situated on the axis of rotation of a swirling stream of infinite extent.

The eddies may either be the wakes of solid bodies placed in the stream, or free eddies, surrounded on all sides by fluid of the outer stream. To make the analysis tractable, the eddies are assumed to be slender, so that their cross-sectional area is a slowly varying function of axial distance. The work is motivated by the desire to examine the possibility of containing one fluid by another with a minimum use of solid boundaries.

'Wake eddies' occur immediately downstream of solid obstacles, such as a cone or cylinder, which induce the main stream to separate. Although viscosity is responsible for separation and hence for the existence of the eddy, and provides the motive force driving the flow within the eddy through shearing action at the eddy boundary, we ignore viscous effects upon the dynamics of the flow away from the boundaries. Hence an inviscid model is assumed in which the viscous shear layer responsible for the effects cited above collapses onto the dividing (free) streamline.

Batchelor (1956*a*) has deduced certain integral conditions upon vorticity which must be satisfied in a region of closed streamlines if the viscosity is non-zero (but arbitrarily small). Presumably, in order for a solution representing such a flow to be the steady limit of a real flow as $Re \to \infty$, it must satisfy these conditions, which then supplement the Euler equations.

For axisymmetric flow, Batchelor showed that these supplementary restrictions select the functional form of the vorticity distribution within the eddy, but not its intensity. (The intensity is presumably related to the details of the bounding shear layers.) In our case, the vorticity distribution selected is that of the Hill's spherical vortex (cf. Lamb 1932). Thus, as may easily be verified, the vorticity within the eddy satisfies the *full* Helmholtz equations. Also, if a constant axial pressure gradient proportional to Re^{-1} is allowed, the velocity distribution within the eddy is an exact solution of the full Navier–Stokes equations.†

We shall not pursue this line of inquiry further here, but shall confine ourselves to the limiting case $Re \to \infty$.

The problem outlined above has important points of contact with that presented by Childress (1966) for slender two-dimensional eddies embedded in a potential flow. This paper may in part be considered a modification of Childress's work to axial symmetry, and to a vortical main stream.

Although the physical situation is more intricate than that treated by Childress, the mathematics involved is in fact much simpler. The simplification realized here is, in its essentials, a feature of well-known properties of axial source distributions. Instead of a non-linear integral equation requiring a numerical solution, we need only deal with a simple non-linear differential equation in order to find the shape of the free streamline. This equation is solved by quadratures, and a solution is found in closed form.

The presence of a solid to induce separation may not be a necessary ingredient in the formation of a self-contained, recirculating eddy embedded in a rotational external stream. We can suppose such an eddy (which we have called the 'free' eddy) to be the product of a 'vortex breakdown'. Again, the analysis is only tractable if it is slender, its slenderness being measured by (say) its fineness ratio.

The characteristic feature of the vortex breakdown phenomena (as described for example by Hall (1966), Benjamin (1962), and observed by Harvey (1962)) is the sudden appearance of a stagnation point on the axis of rotation in a swirling stream, followed by a recirculating eddying region. This region may under some circumstances be closed and self-contained. For example, some of the breakdowns observed by Harvey were closed and elongated.

Although the details of the mechanism responsible for the transition from the primary swirling flow remain obscure, it is, nevertheless, of interest to attempt

† This has been pointed out by Batchelor (1956*a*, p. 187).

an investigation of the structure of the two-celled flows which result after a breakdown.

Some of the photographs of breakdowns produced in cylindrical tubes (cf. Chanaud 1965) show them to be slender, with fineness ratios of the order of 10^{-1}. It is hoped that consideration of slender eddies producing small perturbations will reveal features present in finite disturbance breakdowns as well.

The solutions for free eddies and wake eddies are found by the same method. Therefore, solutions are first carried out for the latter case, and then the slight adaptations required for the application to free eddies are indicated.

In §4, small perturbation equations are presented which apply for arbitrary vorticity distributions in the main stream. Axisymmetric slender body theory for potential flow is then modified in detail for rotational streams for the case of 'wheel flows' which have solid body rotation at upstream infinity. The dynamic pressure on the outside eddy surface is found. The disturbance displays the well-known Taylor non-uniqueness (cf. Squire 1956) but this does not affect the boundary shape to the usual order of approximation in slender body theory.

Within the eddy, the solution for the flow is found by employing approximations of boundary-layer type; and the dynamic pressure on the interior eddy surface is found in §5.

In §6, the pressures at the eddy interface are matched, and an equation for the shape of the free streamline is deduced. The cusped nature of the shape of any eddy free end is discussed. The fact that free ends are cusped is not a consequence of the particular approximations used in this paper. Batchelor (1956*b*, p. 394) has presented a simple argument which shows that any closed eddy bounded by a shear layer must have cusped free ends. Batchelor's argument can be used to show that if the speed just inside the eddy is faster than that outside, the cusp must be re-entrant, rather like a smoke ring which has lost its hole. For then, $\frac{1}{2}\rho(U^2 - V^2) = \text{constant} > 0$ if U is the speed inside and V that outside along the discontinuity surface. Thus, if the eddy is closed it must have a cusp inside on the axis, and a stagnation point outside. If re-entrant cusps are excluded, as in this paper, the speed inside must be less than that outside. If the speeds match, of course, there is no cusp (such is the case, for example, with the Hill's spherical vortex).

The shape of the free streamline is calculated in §7, and a solution away from the cusped region is obtained in closed form. The profiles for both wake and free eddies comprise one-parameter families of curves. Although the profile shapes are in fact insensitive to changes of this parameter, the lengths of free eddies depend strongly upon it, being given by

$$4(1-2\nu)^{\frac{1}{2}} K(\nu) \frac{W_\infty}{\Omega_\infty},$$

where K is the complete elliptic integral, W_∞ the undisturbed mainstream speed, Ω_∞ its angular velocity, and ν the parameter characterizing the problem. Here ν above is restricted to the range $(0, \frac{1}{2})$, so it is predicted that the maximum possible length for a very slender eddy is

$$2\pi(W_\infty/\Omega_\infty).$$

Since ν is related to the strength of the eddy vorticity, a measure of the eddy length permits an estimate of interior vorticity to be made for given conditions of the surrounding stream.

In regard to the question of forces on a free eddy, a remark concerning the relation between viscous effects (ignored in the paper) and the pressure in the surrounding stream should be made. The drag experienced by an eddy in a stream without an ambient pressure gradient is in part an inviscid wave drag (if downstream waves are produced) and in part due to viscous effects. Under such circumstances a stationary eddy cannot be maintained. For the eddy to remain in equilibrium, the drag must be balanced by a thrust exerted by an adverse pressure gradient outside. Thus we expect that in a real fluid, the base flow must decelerate in order to maintain a stationary eddy. A modification to the theory presented here to inviscid eddies embedded in a pressure gradient will be presented by the author elsewhere.

2. Formulation of the problem

It is well known (cf. Squire 1956) that the Euler equations governing steady, incompressible, inviscid, axisymmetric flows may be reduced to the following form:

$$\left.\begin{aligned} &(a)\ \ D^2\Psi \equiv \Psi_{rr} - (1/r)\Psi_r + \Psi_{zz} = r^2H'(\Psi) - FF'(\Psi),\\ &(b)\ \ rv = F(\Psi),\\ &(c)\ \ rw = \Psi_r,\\ &(d)\ \ ru = -\Psi_z,\\ &(e)\ \ p + \tfrac{1}{2}\rho(u^2+v^2+w^2) = \rho H(\Psi). \end{aligned}\right\} \qquad (1)$$

Here u, v, w are, respectively, the radial (r), azimuthal, and axial (z) components of velocity. The pressure is p, the density is ρ, and H and F are arbitrary functions of the meridional stream function Ψ [defined by ($1c;d$)] which are determined providing the flow is known at an initial station.

The azimuthal component of vorticity ζ is given by

$$\zeta = -(1/r)\{r^2H'(\Psi) - FF'(\Psi)\}.$$

Consider an infinite flow field having a representative axial speed W_∞ and angular speed Ω_∞ (about the symmetry axis). Non-dimensionalize (1) by comparing all velocities to W_∞, H to $\frac{1}{2}W_\infty^2$, and pressure ρW_∞^2. Lengths will be compared to a characteristic length L, which will be specified later, F to $L^2\Omega_\infty$ and Ψ to L^2W_∞.

Using the same symbols to denote dimensionless quantities, the non-dimensional form of (1) is:

$$\left.\begin{aligned} &(a)\ \ D^2\Psi = \tfrac{1}{2}r^2H'(\Psi) - \lambda^2FF'(\Psi),\\ &(b)\ \ rv = F(\Psi),\\ &(c)\ \ rw = \Psi_r,\\ &(d)\ \ ru = -\Psi_z,\\ &(e)\ \ p + \tfrac{1}{2}(u^2+v^2+w^2) = \tfrac{1}{2}H(\Psi). \end{aligned}\right\} \qquad (2)$$

The dimensionless swirl parameter $\lambda = L\Omega_\infty/W_\infty$, the inverse Rossby number, is a measure of the relative importance of azimuthal velocity at a distance L from the axis, to through-flow velocity. It is also known as the tangent of the 'swirl angle', which is the helix pitch angle of a particle if it is imagined to move on a helix located at a distance L from the axis.

The portion of the flow determined by streamlines originating at upstream infinity ($z \to -\infty$) will be called the 'outer', or 'base' flow, and quantities pertaining to it will be denoted by the subscript o. In particular, specification of the flow as $z \to -\infty$, which we will assume to be given, determines the functions $H_o(\Psi_o)$ and $F_o(\Psi_o)$.

In addition we consider a bounded region composed of closed streamlines, which we shall variously call the 'inner' region, or the 'eddy', and the flow in it, the 'inner' flow. This region will be governed in general by a different vorticity distribution from the outer flow. Presumably, the vorticity distribution in the inner region is determined by the requirement that the flow be truly steady as

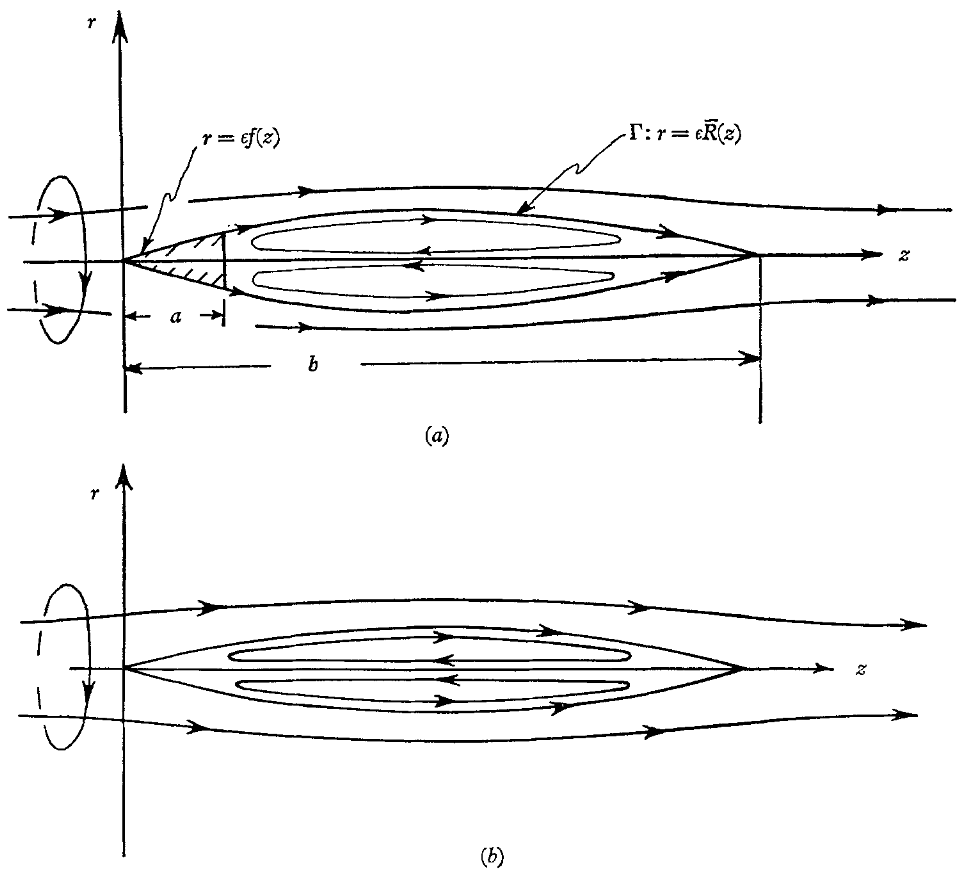

FIGURE 1. Configurations for (*a*) wake eddies, and (*b*) free eddies.

$Re \to \infty$ from finite values, as discussed by Batchelor (1956*a*). Quantities pertaining to this flow will be denoted by the subscript i, e.g. $H_i(\Psi_i)$ and $F_i(\Psi_i)$.

To be consistent with the assumption of axial symmetry, the inner region must be centred upon, and symmetric about, the z-axis. In the problems that we shall consider, the inner region will be the interior of a body of revolution.

We will be basically concerned with two related questions.

The first is the determination of the separated (inviscid) flow engendered by the presence of a solid symmetric obstacle placed on the axis. The obstacle is slender, its shape being given by $r = \epsilon f(z)$, $0 \leqslant z \leqslant a$. Here ϵ is a small parameter, as for example the slope of the solid at $z = a$, so that the disturbance to the outer base flow is small [of $O(\epsilon^2)$]. If the undisturbed base flow is given by Ψ_b, then $\Psi_o = \Psi_b + \psi_o$, where $\Psi_b = O(\epsilon^0)$, and the disturbance to the outer flow ψ_o is $O(\epsilon^2)$. The base flow is disturbed by the solid surface, separates, and then reforms a finite distance downstream leaving behind a recirculating eddy. A free streamline Γ, defined by $r = \epsilon \bar{R}(z)$ separates the eddy from the outer flow in this inviscid model. It has been implicitly assumed here that the eddy region will have a thickness of the same order as the obstacle inducing separation.

Figure 1*a* illustrates the situation being considered. In order for it to hold, the Reynolds numbers both inside and outside the eddy, based upon a typical eddy dimension, must be appropriately large.

The second question concerns the 'free eddy' model for vortex breakdown, and is illustrated figure 1*b*. It is assumed that the eddy begins at $z = 0$, and ends at $z = b$.

3. Boundary conditions

On the dividing streamline, the inner and outer flows must satisfy the kinematic condition

$$\Psi_o = \Psi_i$$

and without loss of generality we may take zero as the common value; and the dynamic requirement,

$$p_i = p_o. \tag{3}$$

Furthermore, the shape of the wake eddy must be such that

$$\left.\begin{aligned} f'(a) &= \bar{R}'(a), \\ f(a) &= \bar{R}(a). \end{aligned}\right\} \tag{4}$$

In the eddy interiors, the assumption that the flow direction near the bounding surface is downstream implies that

$$\Psi_i \leqslant 0,$$

the equality being attained on Γ, on the axis $r = 0$, and on any other curves forming part of the eddy boundary in the meridional plane.

On $r = 0$ in the external flow $\Psi_o(0, z) = 0$.

We require all velocities to be finite for $r < \infty$. Therefore, a stronger statement may be made concerning the behaviour of Ψ_o and Ψ_i near the axis, i.e.

$$\Psi_{o,i} = O(r^2) \quad \text{as} \quad r \to 0.$$

As $r^2 + z^2 \to \infty$, grad $(\Psi_o \to \Psi_b) \to 0$, the given base flow, so for large distances,

$$\text{grad}\ \psi_o \to 0. \tag{5}$$

In order that the tangential velocity remain bounded on $r = 0$, where $\Psi_i = \Psi_o = 0$, F_i and F_o are restrained to have

$$F_i(0) = 0 = F_0(0).$$

Hence, tangential velocities vanish on Γ. Also, since Γ is a streamline, H_o and H_i are (in general, different) constants on Γ, the difference of which we take to be h,

$$H_o(0) - H_i(0) = h.$$

Continuity of pressure is assured if

$$w_o^2[1+\epsilon^2(\bar{R}')^2] - w_i^2[1+\epsilon^2(\bar{R}')^2] = h. \tag{6}$$

Since the base flow is only slightly disturbed, write

$$w_o(\epsilon\bar{R}, z) = W_o(z) + w^*,$$

where

$$W_o(z) = w_b(\epsilon\bar{R}, z)$$

and $w_b(\epsilon\bar{R}, z)$ is the axial velocity that would be attained at $r = \epsilon\bar{R}$ in the undisturbed base flow, and w^* is the disturbance to the outer flow caused by the presence of the eddy. But $w^* \ll W_o$ by hypothesis, so, on ignoring error terms consistent with the accuracy sought here, (6) may be approximated as

$$W_o^2 + 2W_o w^* - w_i^2 = h. \tag{7}$$

This is the form of (3) that is convenient to apply.

4. The outer flow

The problem for the outer flow for wake eddies is identical to that of finding the disturbance to Ψ_b caused by a slender body of shape $r = \epsilon R(z)$, where

$$R(z) = \begin{Bmatrix} 0 & (0 \leqslant z, z \geqslant b), \\ f(z) & (0 \leqslant z \leqslant a), \\ \bar{R}(z) & (a \leqslant z \leqslant b). \end{Bmatrix} \tag{8}$$

Analogous remarks apply to the case of the free eddy, in which case $a \to 0$ in (8) and f does not appear.

$R^2(z)$ will be assumed to have a continuous second derivative in $(0, b)$.

It was previously pointed out that the disturbance ψ_o to Ψ_b is of second order in ϵ. Assume $H'(\Psi)$ and $FF'(\Psi)$ to have second derivatives. Then

$$H'(\Psi) = H'(\Psi_b) + H''(\Psi_b)\,\psi + O(\epsilon^4)$$

and

$$FF'(\Psi) = FF'(\Psi_b) + \{[F'(\Psi_b)]^2 + FF''(\Psi_b)\}\,\psi + O(\epsilon^4).$$

(As this section involves only the outer flow, we shall drop the subscript o.)

Introducing these expansions into (2), and remembering that Ψ_b itself satisfies (2), we find that on ignoring a relative error of order ϵ^2,

$$D^2\psi = [\tfrac{1}{2}r^2 m(r,z) - n(r,z)]\,\psi, \tag{9}$$

where

$$m = H''[\Psi_b(r,z)]$$

and

$$n = \tfrac{1}{2}\lambda^2\,[F^2(\Psi_b)]''.$$

Equation (9) is linear, but cannot be solved until Ψ_b and F (hence H, thence m and n) are prescribed.

The simplest non-trivial choice leading to a swirling outer flow is wheel flow, in which

$$\Psi_b = \tfrac{1}{2}r^2 \quad \text{and} \quad F(\Psi) = \Psi.$$

Wheel flow is the superposition of a uniform axial stream with a solid body rotation. With the Ψ_b and F_o for wheel flow, (9) assumes the form

$$D^2\psi + \lambda^2\psi = 0, \tag{10}$$

and the equation is exact. That is, in wheel flow, all of the error terms in (9) vanish. An extensive literature on the perturbation of wheel flows by solid bodies exists, the reader is referred to Squire's (1956) review for examples.

Because of its simplicity, we shall carry out the analysis for wheel-flow. It should be emphasized, however, that other choices are also tractable analytically, although the results are more complicated.

(In regard to our model for vortex breakdown, there may be disadvantages in the choice of this simple outer flow. It is completely stable, and therefore is not likely to engender a breakdown and hence a secondary vortex. In each of the experimental cases cited, the primary flow possessed azimuthal vorticity and (often) an axial pressure gradient, neither of which is present in simple wheel flow. Our analysis can be applied to generalized wheel flows, which have pressure gradients, so this latter objection is not serious. Also, it seems to be true that observed outer flows are reasonably well approximated by wheel flow, at least over the immediate region of the secondary flow. Thus, although the inception of the eddy may be due to features of the outer flow lacking in wheel flow, the structure of the eddy may possibly be well represented by an interaction with an external wheel flow.)

The boundary condition on the surface $r = \epsilon R$ is

$$\Psi = \Psi_b + \psi = 0.$$

But, on $r = \epsilon R$, $\Psi_b = \frac{1}{2}\epsilon^2 R^2$, so

$$\psi = -\tfrac{1}{2}\epsilon^2 R^2 \tag{11}$$

on $r = \epsilon R$.

As in slender body theory, we may transfer the boundary conditions (11) from the body surface to the axis $r = 0$, with error of $O\{\epsilon^4 \log(1/\epsilon)\}$. At infinity, the boundary condition (5) applies.

We shall now solve for ψ by operational methods which amount to a modification of the approach of Adams & Sears (1953) towards slender body theory to the case where swirl exists.

Apply the Fourier transform,

$$\bar{g}(r;\omega) = \int_{-\infty}^{\infty} g(r,z)\, \epsilon^{-i\omega z} dz,$$

to (10) to obtain

$$\bar{\psi}_{rr} - (1/r)\,\bar{\psi}_r + (\lambda^2 - \omega^2)\,\bar{\psi} = 0. \tag{12}$$

Equation (12) has solution

$$\bar{\psi}(r;\omega) = (\epsilon^2/2\pi)\, r\{A(\omega)\, I_1(r\Omega) + B(\omega)\, K_1(r\Omega)\},$$

where I_1 and K_1 are modified Bessel functions, and

$$\Omega(\omega) = (\omega^2 - \lambda^2)^{\frac{1}{2}}.$$

Let

$$\omega = \sigma + i\tau,$$

then for $|\sigma| > \lambda$ on $\tau = 0$, Ω is real. Branch cuts may be chosen such that $\Omega > 0$ for $|\sigma| > \lambda$ on the real ω-axis; or alternatively, in such a way that Ω changes sign for $|\sigma| > \lambda$ as σ itself changes sign. These latter branches are not acceptable, since K_1 diverges for large negative Ω, and I_1 as $\mathrm{Re}\,\Omega \rightarrow +\infty$. Therefore, a branch is chosen with $\Omega > 0$ for $|\sigma| > 0$ on $\tau = 0$. Two suitable choices for cuts are given in the appendix.

Denote $\pi\epsilon^2 R^2(z)$ by the symbol $\epsilon^2 S(z)$, which then represents the cross-sectional area of the body at z, and let its transform be $\epsilon^2 \bar{S}(\omega)$. Then the transform of (11) is

$$\bar{\psi}(0;\omega) = -(\epsilon^2/2\pi)\,\bar{S}(\omega). \tag{13}$$

Since $I_1(x) \rightarrow \infty$ for $\mathrm{Re}\,x \rightarrow \infty$, then for ω real,

$$A(\sigma) = 0 \quad \text{for} \quad |\sigma| > \lambda. \tag{14}$$

Also
$$rK_1(r\Omega) = (1/\Omega)\,[1 + O(r^2\Omega^2 \log r^2\Omega^2)],$$
and
$$rI_1(r\Omega) = O(r^2\Omega) \quad \text{as} \quad r \rightarrow 0.$$
Hence,
$$B(\omega) = -\Omega(\omega)\,\bar{S}$$
in order to satisfy (13).

Therefore, the solution for $\bar{\psi}$ is

$$\bar{\psi} = (\epsilon^2/2\pi)\,r\{A(\omega)\,I_1(r\Omega) - \Omega\bar{S}(\omega)\,K_1(r\Omega)\}$$

so that the disturbance axial velocity, w^*, has transform

$$\bar{w}^* = (1/r)\,\bar{\psi}_r = (\epsilon^2/2\pi)\,\{A\,\Omega I_0(r\Omega) + \Omega^2\bar{S}(\omega)K_0(r\Omega)\}. \tag{15}$$

The solution (15) is not unique, since the boundary conditions fail to determine $A(\omega)$ completely, instead specifying it only over a portion of the real axis by (14). This is a manifestation of the non-uniqueness in rotating flows past bodies first noted by G. I. Taylor (cf. Squire 1956) and further discussed by Fraenkel (1956), and by Stewartson (1958). The non-uniqueness has yet to be resolved, but it is clear that additional conditions must be placed upon the flow to render it unique. One possibility, considered by Fraenkel, is to confine the flow with a tube but this fails when λ is increased above a critical value. Fraenkel and Stewartson have proposed that there should be no upstream waves, a condition suggested by a transient analysis. We shall not speculate as to what are the correct additional conditions required, since the non-uniqueness will be shown not to affect the interaction with the eddy to our order of approximation. We only note that Stewartson's criterion will yield a unique outer flow in our case.

The inversion of (15) is carried out in the appendix where it is shown that

$$\begin{aligned} w^* = -\frac{\epsilon^2}{4\pi}\Bigg\{&\int_0^b [S''(t) + \lambda^2 S(t)]\,\frac{\cos[\lambda(r^2+(z-t)^2)^{\frac{1}{2}}]}{(r^2+(z-t)^2)^{\frac{1}{2}}}\,dt \\ &-\frac{\lambda^2}{\pi}\int_{-1}^{1} [A_1(\lambda t)\sin\lambda zt + A_2(\lambda t)\cos\lambda zt]\,(1-t^2)^{\frac{1}{2}}\,J_0[r\lambda(1-t^2)^{\frac{1}{2}}]\,dt, \end{aligned} \tag{16}$$

where $A_1 = \mathrm{Re}\,A(\omega)$, $A_2 = \mathrm{Im}\,A(\omega)$. In order for w^* to be real, A and S must be related by

$$\begin{aligned} &\int_0^b [S''(t) + \lambda^2 S(t)]\,\frac{\sin[\lambda(r^2+(z-t)^2)^{\frac{1}{2}}]}{(r^2+(z-t)^2)^{\frac{1}{2}}}\,dt \\ &\quad + \frac{\lambda^2}{\pi}\int_{-1}^{1} [A_1(\lambda t)\cos\lambda tz - A_2(\lambda t)\sin\lambda tz]\,(1-t^2)^{\frac{1}{2}}\,J_0[r\lambda(1-t^2)^{\frac{1}{2}}]\,dt = 0. \end{aligned}$$

The formula (16) generalizes that encountered in potential slender body theory (cf. Adams & Sears 1953), to which it reduces upon setting $\lambda = 0$.

It is further shown by some manipulation in the appendix, and by carrying over results for the irrotational case which may be found in Ward (1955, p. 189), that on and near the surface $r = \epsilon R(z)$ $(0 < z < b)$,

$$w^* = \frac{\epsilon^2}{2\pi}\left\{[S''(z)+\lambda^2 S(z)]\log \tfrac{1}{2}r + \frac{1}{2}\int_{z+\delta}^{b+0} \log(t-z)\, d(S''+\lambda^2 S) - \frac{1}{2}\int_{0-}^{z-\delta} \log(z-t)\, d(S''+\lambda^2 S)\right\} + O[(\lambda^2+1)\epsilon^2].$$

Here the integrals are interpreted in the Stieltjes sense, and the limit $\delta \to 0$ is to be taken.

Except possibly in the vicinity of the leading and trailing ends

$$w^* = (\epsilon^2/2\pi)[S''(z)+\lambda^2 S(z)]\log \tfrac{1}{2}r + O[(\lambda^2+1)\epsilon^2]. \tag{17}$$

The error in this expression is $O[\epsilon^2 S''(0+)\log z] + O[\epsilon^2 S''(b-)\log(b-z)]$ so that if $S''(0+) = S''(b-) = 0$, (17) holds uniformly on the eddy surface. It will be shown later that a consistent solution for the eddy shape near free ends requires S'' to vanish there. In the case of a wake eddy $S''(0+)$ occurs on the solid surface, so it is consistent to assume that (17) holds uniformly on Γ, the free streamline.

Ignoring the error terms in (17), on Γ

$$w^* = \epsilon^2(\sigma''+\lambda^2\sigma)\log \epsilon^2\sigma, \tag{18}$$

where we have put $\sigma(z) = (1/4\pi)S(z)$.

From (18), the perturbation axial velocity is of order $\epsilon^2\log(1/\epsilon^2)$. Notice that the undetermined portion of the solution for w^*, i.e. the part involving the unknown quantities A_1, A_2, is $O(\epsilon^2)$, as are the error terms in (17). Thus to lowest order they do not affect the perturbation velocity on $r = \epsilon R(z)$. It should be kept in mind, however, that in practical situations ϵ^2 terms are comparable to $\epsilon^2\log(1/\epsilon)$ terms even for ϵ fairly small, so this error can be appreciable. Also notice that the approximation requires $\epsilon\lambda$ to be small, as well as ϵ itself. In terms of physical variables, this demands that $r_o\Omega_\infty/W_\infty \ll 1$, where r_o is the maximum radial extent of Γ. This, in effect, is a requirement that the perturbation azimuthal velocity should be small. At the interface, v is perturbed from its undisturbed value $r_o\Omega_\infty$ to zero. In the experimental run made by Chanaud (1965) referred to in the discussion to follow, λ based upon eddy length, was about 2, for an $\epsilon \doteqdot 0{\cdot}13$.

5. The inner flow

From §2, Ψ_i satisfies the equation

$$D^2\Psi_i = \tfrac{1}{2}r^2 H_i'(\Psi_i) - \lambda^2 F_i F_i'(\Psi_i). \tag{19}$$

(Only the inner flow is considered in this section, so we shall drop the subscript.)

As Batchelor (1956*a*) has shown, for axisymmetric regions of closed streamlines in steady flow, there can be no distributed axial vorticity as $Re \to \infty$ from finite values. Azimuthal swirl is permitted, but only of *potential* type, i.e. $rv =$ constant. Since the axis $r = 0$ may be approached within our eddy, we must require this constant to be zero. Thus there is no azimuthal swirl. Batchelor further shows that, if swirl is absent, the single (azimuthal) component of vorticity remaining must be proportional to r alone, so that $H_i'(\Psi_i) =$ constant.

Recognizing that the interior vorticity may be large, we write (in view of the above remarks)

$$H'(\Psi) = 2\beta/\epsilon^2. \tag{20}$$

Introduce the magnified co-ordinate

$$\eta = \epsilon^{-1}r$$

into (19) to obtain

$$\Psi_{\eta\eta} - (1/\eta)\Psi_\eta + \epsilon^2\Psi_{zz} = \epsilon^2\beta\eta^2.$$

Since $\Psi = O(\epsilon^2)$ (§3), put

$$\Psi(r,z) = \epsilon^2\psi(\eta,z).$$

Then in the eddy we have

$$\psi_{\eta\eta} - (1/\eta)\psi_\eta - \beta\eta^2 = -\epsilon^2\psi_{zz}. \tag{21}$$

It is expected that ψ_{zz} will be bounded, except possibly in the neighbourhood of the eddy ends. Thus, we can ignore the right-hand side of (21) except possibly at the eddy ends, where a 'boundary-layer' type of behaviour may require the full equation (21) for solution. We shall not consider this complication, but merely point out that it is likely that such singular regions can be handled by the same method used by Childress (1966) in the two-dimensional case. An argument will be advanced later in support of the view that such singular zones will have at most a weak effect upon the solution for σ at an eddy free end, although they govern the flow locally.

Since Γ is given by the curve $\eta = R(z)$, the appropriate boundary conditions on ψ are

$$\psi(0;z) = \psi(R;z) = 0 \tag{22}$$

and $\psi \leqslant 0$ for all η. Thus z enters the solution only as a parameter through the condition at $\eta = R$. It is not possible to satisfy the boundary condition $\psi = 0$ at $z = a$, hence the above remarks about singular zones.

The solution for w appropriate to (21) and (22) is

$$w = \tfrac{1}{4}\beta[2\eta^2 - R^2]. \tag{23}$$

Here $\beta \geqslant 0$ in order to keep ψ non-positive in the eddy as required. On Γ

$$w = \tfrac{1}{4}\beta R^2 = \beta\sigma(z). \tag{24}$$

6. Pressure match

Certain deductions about the flow in the eddy can now be made from the pressure condition (7), which will in fact provide the equation determining the eddy shape $S(z)$.

From (7) and (24)

$$1 + 2w^* - \beta^2\sigma^2 = h \tag{25}$$

since $W_0(z) = 1$.

It is clear from (25) that it is not possible to have $\beta^2 = O(1)$ as $\epsilon \to 0$. If it were, then σ would be a constant. This violates the assumptions concerning slenderness at the free end or ends of the eddy, where R would have to fall rapidly to zero. Thus $\beta^2 = O\{\epsilon^2 \log(1/\epsilon^2)\}$, and we therefore take

$$\beta^2 = 2\alpha^2\epsilon^2 \log(1/\epsilon^2) \tag{26}$$

and, in addition,

$$h = 1 - 2k\epsilon^2 \log(1/\epsilon^2), \tag{27}$$

where k and α^2 are assumed to be independent of ϵ as $\epsilon \to 0$.

Continuity of pressure is then assured providing

$$w^* = \epsilon^2 \log(1/\epsilon^2)[\alpha^2\sigma^2 - k],$$

or

$$[\sigma''(z) + \lambda^2\sigma(z)] \log\left(\frac{1}{\epsilon^2\sigma}\right) = \log\frac{1}{\epsilon^2}[k - \alpha^2\sigma^2]. \tag{28}$$

Since w^* is in error by $O(\epsilon^2)$, equation (28) has a relative error of $O(1/\log \epsilon^2\sigma)$.

A solution for σ will be found in the next section. First, however, we summarize the results of this section.

As $\sigma \to 0$ (as it must at a free end) equation (28) requires $\sigma''(z) \to 0$ as well. Hence the free end of an eddy has a cusped shape (the consistency of the remarks following (17) is hereby demonstrated). It has been known for some time that a general requirement for the 'free end' of an eddy bounded by a vortex sheet is that it must be cusped (see §1).

Since the approximate solution for w_i vanishes as $\sigma \to 0$, as does the exact solution inside the eddy, it is unlikely that the approximations made in the inner flow will much affect the solution for the cusp. The axial extent of the cusp area here is small, since the logarithm is a slowly varying function. In fact, the cusp 'length' vanishes exponentially as $\epsilon \to 0$, a fact perhaps connected either with the application of linearized boundary conditions to the outer flow, or the neglect of $O(\epsilon^2)$ terms.

Since $\beta = O[\epsilon\{\log(1/\epsilon)\}^{\frac{1}{2}}]$, the inner velocity is small of this order, but the inner vorticity is large, being of $O[\{\log(1/\epsilon)\}^{\frac{1}{2}}]$.

7. Eddy shape

Equation (28) governs σ, and the appropriate boundary conditions have already been given in §3, (4).

To integrate (28) let

$$d\sigma/dz = \Phi,$$

then $\sigma'' = \Phi(d\Phi/d\sigma)$, and hence

$$\Phi^2 = -\lambda^2\sigma^2 + 2\log\frac{1}{\epsilon^2}\left\{-\frac{k}{\epsilon^2}\operatorname{li}(\epsilon^2\sigma) + \frac{\alpha^2}{\epsilon^6}\operatorname{li}(\epsilon^6\sigma^3)\right\},$$

where the limits of integration have been chosen to satisfy the boundary condition $\Phi = 0$ at $\sigma = 0$. Here $\operatorname{li}(x)$ is the logarithmic integral

$$\operatorname{li}(x) = \int_0^x \frac{dt}{\log t},$$

a function which is tabulated (cf. Abramowitz & Stegun 1964).

But as $x \to 0$, li(x) has the asymptotic expansion

$$\mathrm{li}(x) \sim -\frac{x}{\log(1/x)}\left\{1 - \frac{1}{\log(1/x)} + \ldots\right\}.$$

Since our original equation (28) has error of

$$O\left[\frac{1}{\log(1/\epsilon^2\sigma)}\right],$$

it is consistent with our accuracy to replace li(x) by $-x/\log(1/x)$ wherever it occurs.

Thus,
$$\frac{d\sigma}{dz} = \pm\left[-\lambda^2\sigma^2 + \frac{2\log(1/\epsilon^2)}{\log(1/\epsilon^2\sigma)}\left(k\sigma - \frac{\alpha^2\sigma^3}{3}\right)\right]^{\frac{1}{2}}. \tag{29}$$

The upper sign clearly holds for $z \leqslant z_o$, where z_o is defined as that value of z for which σ reaches a maximum or $\sigma \leqslant \sigma(z_o)$ in the interval (a, b). The lower sign then holds for $z \geqslant z_o$. At z_o, of course, $\sigma' = 0$.

Furthermore,
$$\frac{\log(1/\epsilon^2)}{\log(1/\epsilon^2\sigma)} = 1 - \frac{\log\sigma}{\log\epsilon^2\sigma}$$

and this deviates from unity only for $\sigma \sim \epsilon^2$.

Accordingly, away from the cusp ($\sigma = 0$), as $\epsilon \to 0$ we have

$$\sigma' = \pm[2k\sigma - \lambda^2\sigma^2 - \tfrac{2}{3}\alpha^2\sigma^3]^{\frac{1}{2}}. \tag{30}$$

We treat this case first, and then return to consider conditions near the cusp, where σ is not large compared to ϵ^2.

At $z = a$ in the wake eddy, we have the additional initial data

$$\left\{\begin{aligned} &\sigma(a) = \tfrac{1}{4}f^2(a), \\ &\sigma'(a) = \tfrac{1}{2}f(a)f'(a) = \left[+\tfrac{1}{2}kf^2(a) - \tfrac{1}{16}\lambda^2 f^4(a) - \frac{2\alpha^2}{3}\left[\frac{f^2(a)}{4}\right]^3\right]^{\frac{1}{2}}. \end{aligned}\right\} \tag{31}$$

Equation (30) may be integrated to give $\sigma(z)$. The equation only has relevance for those positive σ for which

$$P(\sigma; \alpha, k, \lambda) \equiv 2k\sigma - \lambda^2\sigma^2 - \tfrac{2}{3}\alpha^2\sigma^3 \geqslant 0.$$

Now it is useful to be specific about the definition of ϵ. It is convenient, for *both* wake and free eddies, to take ϵ as the profile fineness ratio, maximum diameter divided by total length.

With this definition of ϵ, the maximum radius occurs when $R = 1$, or $\sigma = \frac{1}{4}$, which we suppose occurs for $z = z_o$. At this point, $\sigma' = 0$.

Thus the only non-negative zeros of P occur at $\sigma = 0$, as already noted, and at $\sigma(z_o) = \frac{1}{4}$. Outside of this range, either σ or P (or both) are negative. Since $P(\frac{1}{4}; \alpha, k, \lambda) = 0$, k may be eliminated in favour of α and λ, i.e.

$$k = \tfrac{1}{8}\lambda^2 + \tfrac{1}{48}\alpha^2, \tag{32}$$

which shows that k must be positive, and

$$P(\sigma; \alpha, \lambda) = \sigma[\tfrac{1}{4} - \sigma][\lambda^2 + \tfrac{2}{3}\alpha^2(\tfrac{1}{4} + \sigma)].$$

Thus, σ increases from its initial value $\sigma(a) = \frac{1}{4}f^2(a)$ (or zero in the free eddy, where $a = 0$, $f(a) = 0$) to its maximum $\sigma = \frac{1}{4}$, occurring at $z = z_o$, and then decreases to zero again at $z = b$.

In view of these remarks, the solution may be written in terms of $F(\Phi|\nu)$, the elliptic integral of the first kind and the Jacobian elliptic function cn (x) (Abramowitz & Stegun 1964):

$$\left.\begin{aligned} R(z) &= \text{cn}\left[\frac{\lambda(z-z_o)}{2(1-2\nu)^{\frac{1}{2}}}\middle|\nu\right] \quad (a < z < b), \\ z_o &= a+(2/\lambda)(1-2\nu)^{\frac{1}{2}}F[\cos^{-1}f(a)|\nu], \\ \nu &= \frac{1}{2}\left(\frac{\alpha^2}{\alpha^2+3\lambda^2}\right). \end{aligned}\right\} \quad (33)$$

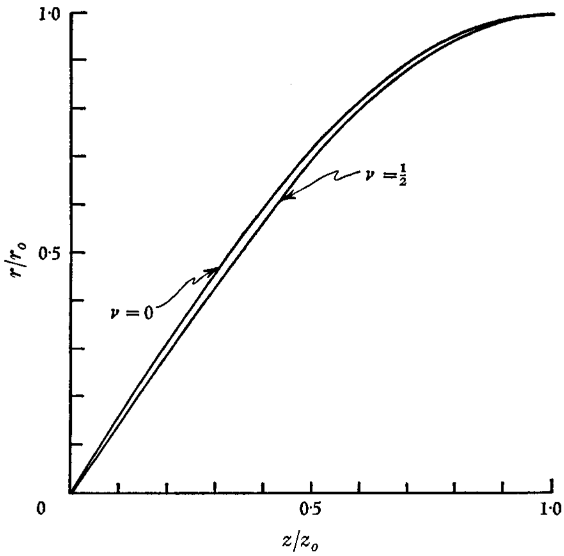

FIGURE 2. Normalized eddy shape function, here $r_o = r(z_o)$ is the maximum radial exten of the eddy. This function may be used to construct the shape of either free or wake eddies

From its definition, $$0 \leqslant \nu \leqslant \tfrac{1}{2},$$
and the event $\nu = \frac{1}{2}$ occurs only if $\lambda = 0$ (potential outer flow). As $\lambda \to 0$

$$\lim_{\lambda\to 0}\frac{2(1-2\nu)^{\frac{1}{2}}}{\lambda} = \frac{2\sqrt{3}}{\alpha}.$$

Plots of z/z_o *vs.* R are shown in figure 2, for $\nu = 0$ and $\nu = \frac{1}{2}$. All other curves for intermediate ν lie between these two.

Notice that the initial condition (31) for wake eddies does not directly enter into the form of the solution. Its effect is to specify the parameter α^2 in terms of λ. Therefore, for a given outer flow (λ), the problem for slender wake eddies appears to be uniquely determined by the geometry of the solid at the separation point, providing that the maximum radius of the separated region can be observed. For then the boundary condition (31) together with (32) requires

$$\alpha^2 = \left(\frac{4[f'(a)]^2}{1-f^2(a)} - \lambda^2\right)\frac{6}{1+f^2(a)}$$

and knowledge of α^2 provides ν, hence the shape and total eddy length, i.e.

$$b = z_o - a + (2/\lambda)(1-2\nu)^{\frac{1}{2}}K(\nu). \tag{34}$$

Similar results hold for the free eddy. If, in (33), we put $a = f(a) = 0$, the solution for this case obtains. Free eddies are therefore symmetric in shape about the station $z = z_o$, and have total length $b = 2z_o$.

Since
$$z_o = (2/\lambda)(1-2\nu)^{\frac{1}{2}}K(\nu),$$
where $K(\nu) = F(\frac{1}{2}\pi|\nu)$ is the complete elliptic integral, we have as total *dimensional* length l of the free eddy

$$l \equiv 2z_o L = 4(1-2\nu)^{\frac{1}{2}}K(\nu)(W_\infty/\Omega_\infty). \tag{35}$$

This predicts that free eddies with $\epsilon \ll 1$ have a maximum length,

$$l_{\max} = 2\pi L_s,$$

where L_s is the 'swirl length', W_∞/Ω_∞. A similar, but more complicated, formula can be written down for the maximum length of wake eddies.

Notice that a measurement of the total length of a free eddy establishes ν via (35), if the external flow (λ) is known. This fixes the vorticity inside the eddy, as α^2 may be found from ν from (33).

If, in addition to l, the maximum diameter (hence ϵ) of the eddy may be measured, the complete shape and hence flow details inside and outside may be calculated.

8. The cusp

So far, solutions have been obtained which exclude the cusp region. In this section, an attempt is made to fill this gap partially.

The difficulty encountered so far in obtaining the closed solution (33) is that the expansion of $\sigma'(z)$ for small ϵ is not uniformly valid. It fails as σ tends to zero, in particular, where $\sigma = O(\epsilon^2)$, which occurs near a free end. This difficulty does not arise at a non-free end, since the solid forming the initial stretch of the free streamline has $\sigma = O(1)$ there.

Here we shall examine the behaviour of the solution in the immediate neighbourhood of a cusp. More precisely, we look at regions for which $R = O(\epsilon)$.

From (29), with $\gamma = +k\log(1/\epsilon^2)$, we have

$$\left(\frac{dR}{dz}\right) \sim \pm\left(\frac{2\gamma}{\log(2/R)}\right)^{\frac{1}{2}} \quad \text{as} \quad R \to 0.$$

This shows that swirl of the outer flow does not affect the shape at the cusp (since λ does not appear), in fact in accord with physical intuition.

The sign choice above depends upon whether the free end being considered is on the upstream or downstream side of the eddy. For definiteness, choose the upper sign, corresponding to an upstream free end.

The cusp solution with $R = 0$ at $z = 0$ is

$$\gamma^{\frac{1}{2}}z = R\left(\log\frac{2}{R}\right)^{\frac{1}{2}} + \sqrt{\pi}\,\mathrm{erfc}\left[\left(\log\frac{2}{R}\right)^{\frac{1}{2}}\right],$$

where erfc (x) is the complementary error function.

9. Discussion

To test the relevance of the present analysis, experimental information about the internal motions in eddies is required. Such experiments have yet to be performed on vortex breakdown flows, since introduction of conventional probes destroys the eddy. Various new and sophisticated methods of flow measurement are currently under development which may be able to circumvent this difficulty. It remains, however, a challenging piece of work.

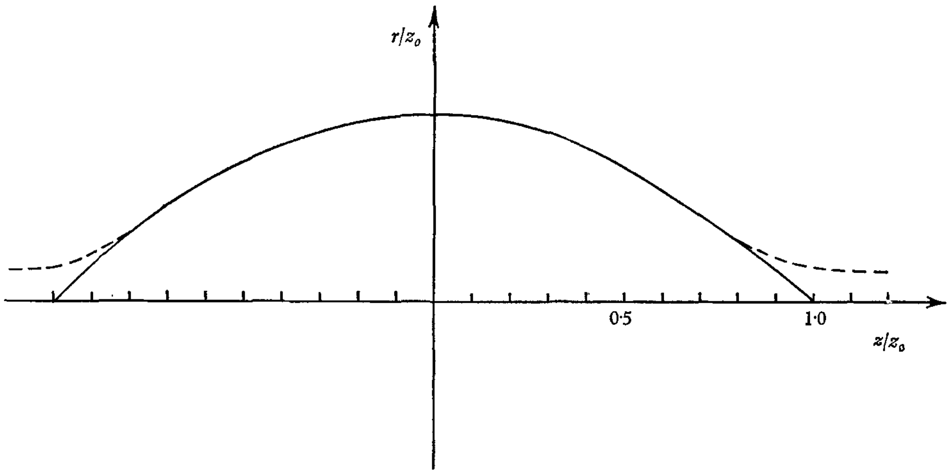

FIGURE 3. Eddy with fineness ratio $\epsilon = 0{\cdot}5$, and $\nu = 0$. Curve for $\nu = \frac{1}{2}$ lies directly below. Dotted segments are drawn to illustrate the way in which a length for the eddy was established from Harvey's photograph.

At the moment, the only available 'data' are photographs of the outlines of vortex breakdown eddies taken by various workers, notably Harvey (1962) and Chanaud (1965). In figures 3, 4 and 5, the predicted shapes for fineness ratios 0·5, 0·53 and 0·13 are plotted. The theory is applicable only when $\epsilon \ll 1$, but the choice of numbers here was made to correspond roughly to photographs appearing, respectively, in figures 3 and 4 of Harvey's paper, and figure 4(*a*) of Chanaud's paper. [The end points of the eddies in the photographs cited are somewhat indistinct (and suggestively cusp shaped).] In order to establish fineness ratios and thereby fix the corresponding theoretical shapes, a choice had to be made as to where the eddies began and ended. We took these to be where the entering dye or smoke filament began noticeably to increase in diameter. The half-length was then taken to be from this point to the point of maximum diameter, and the latter dimension was also measured.

Notice that the shapes of the predicted curves and those appearing in the photographs are very similar. In fact, if the curves in figures 3, 4 and 5 are drawn to the same scale as their photographed counterparts and overlaid on them, the agreement is striking. Of course, many functions other than (33) also describe the rather undistinguished eddy shapes, so that the agreement may be only circumstantial, and no serious conclusions should be drawn from the favourable comparison. This is particularly true in the light of the inappropriateness of the theory for such sizeable ϵ, neglect of terms $O(\epsilon^2)$ compared to those $O\{\epsilon^2 \log(1/\epsilon)\}$ is not justified.

Notice that (35) shows that, for a given L_s, the stagnant eddy is the longest, while the most vigorous is the shortest. Irrotational outer flow is an exceptional case for which the eddy length is not determined by the interior vorticity. In the latter case, the problem has no natural length scale to which eddy length may be compared, hence eddy lengths are not calculable from this theory.

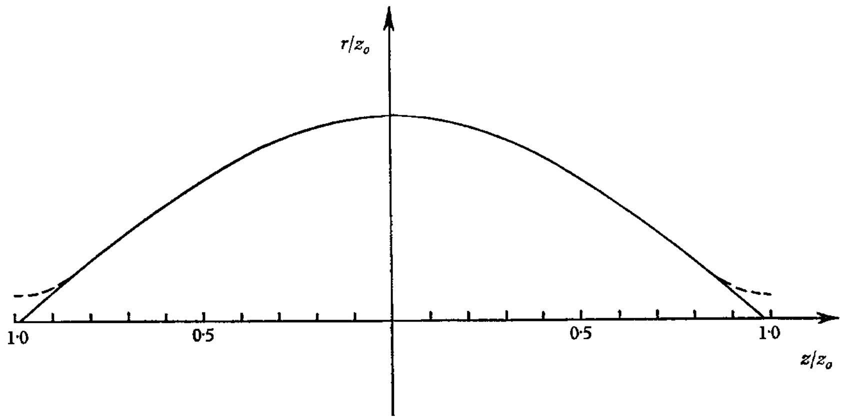

FIGURE 4. Eddy with fineness ratio $\epsilon = 0{\cdot}53$, $\nu = 0$.

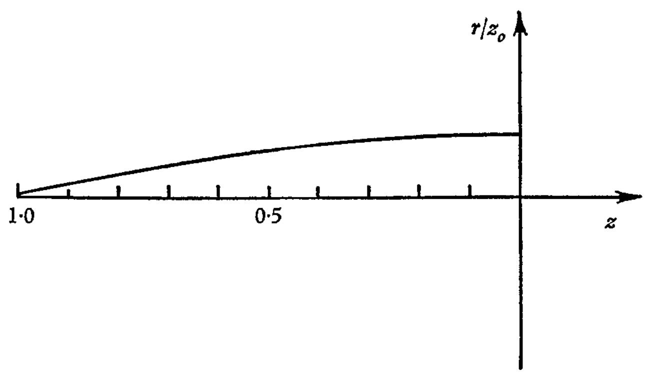

FIGURE 5. Eddy with fineness ratio $\epsilon = 0{\cdot}13$, $\nu = 0$, forward half only.

It is clear that the density of the fluid in the eddy can differ from that in the main stream without affecting the analysis. If surface tension effects may be ignored (as with gases), and if the speed inside the eddy remains less than that outside (so the eddy does not run down by doing work on the flow outside through the viscous shear layers), the analysis may describe a flow involving two different fluids.

The author is indebted to Prof. F. K. Moore for several discussions culminating in the formulation of this problem, and for his helpful criticism of a draft of the paper. The author would also like to express his gratitude to Dr R. C. Chanaud for the loan of several unpublished photographs of vortex breakdowns obtained in his researches. Finally, helpful comments of the referees are appreciated. This work was sponsored by a grant from NASA, technically monitored by the Lewis Research Center.

Appendix

Let

$$I_1 = \frac{1}{2\pi}\int_{-\infty}^{\infty} (\omega^2-\lambda^2)\,\bar{S}(\omega)\,K_o(r\Omega)\,e^{i\omega z}d\omega.$$

Then the inversion of (15) is

$$w^* = \frac{\epsilon^2}{4\pi^2}\int_{-\infty}^{\infty} A(\omega)\,\Omega(\omega)\,I_o(r\Omega)\,e^{i\omega z}d\omega + \frac{\epsilon^2}{2\pi}I_1, \tag{A 1}$$

and we shall evaluate I_1 by means of a contour integral to arrive at (16).

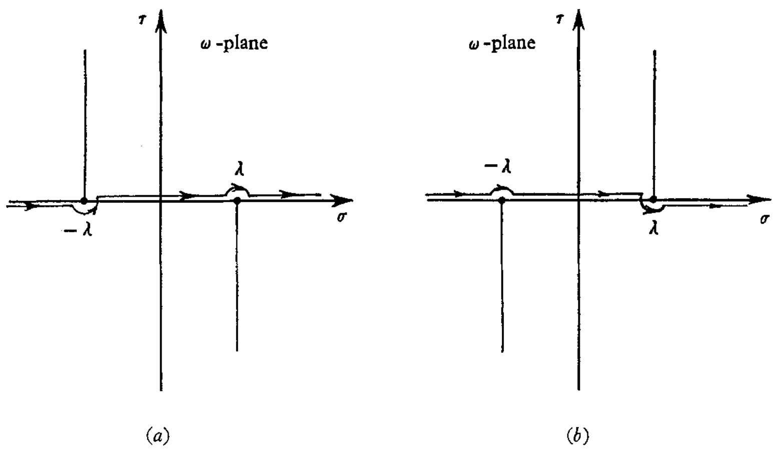

FIGURE 6. The ω-plane showing two acceptable branches for Ω and corresponding integration paths for $G(z)$. (a) Cuts for Ω_1. (b) Cuts defining Ω_2.

By the convolution theorem and the definition of $\bar{S}(\omega)$,

$$I_1 = -\int_{-\infty}^{\infty} s(t)\,G(z-t)\,dt, \quad s(z) = S''(z)+\lambda^2 S(z),$$

$$G(z) = \frac{1}{2\pi}\int_{-\infty}^{\infty} K_o(r\Omega)\,e^{i\omega z}d\omega.$$

Cut arrangements for two acceptable branches for Ω (cf. discussion following (12)) are shown above in figures (6a) and (6b), together with the associated paths of integration for $G(z)$. On both branches Ω_1 and Ω_2 we have $\Omega(\sigma) = \Omega(-\sigma)$. For definiteness, we choose branch $\Omega_1(\omega)$.

Since Ω_1 is even in σ for ω real, we can write

$$G = \frac{1}{\pi}\int_0^{\infty} K_o(r\Omega_1)\cos\sigma z\,d\sigma.$$

Consider the two functions

$$Q_{\pm}(z) = \frac{1}{\pi}\int_0^\infty K_o[r\Omega_1(\sigma)]\, e^{\pm i\sigma z} d\sigma$$

for z positive.

By completing the contour around the boundaries of the first quadrant in the ω-plane,

$$Q_+ = \frac{i}{\pi}\int_0^\infty K_0[ir(\tau^2+\lambda^2)^{\frac{1}{2}}]\, e^{-\tau z} d\tau. \tag{A 2}$$

To evaluate Q_-, note that Ω_1 and Ω_2 are related for $\operatorname{Im}\omega = 0$ by the following: $\Omega_1(\sigma) = \Omega_2(\sigma)$, for $|\sigma| > \lambda$; and $\Omega_1(\sigma) = i(\lambda^2-\sigma^2)^{\frac{1}{2}} = -\Omega_2(\sigma)$, for $|\sigma| < \lambda$.

Therefore (cf. Abramowitz & Stegun 1964, §9.6),

$$K_o(r\Omega_1) = K_o(r\Omega_2) - \pi i J_o[r(\lambda^2-\sigma^2)^{\frac{1}{2}}] \quad \text{for} \quad |\sigma| < \lambda,$$
$$K_o(r\Omega_1) = K_o(r\Omega_2) \quad \text{for} \quad |\sigma| > \lambda,$$

so that Q_- may be expressed as

$$Q_- = \frac{1}{\pi}\int_0^\infty K_o(r\Omega_2)\, e^{-i\sigma z} d\sigma - i\int_0^\lambda J_o[r(\lambda^2-\sigma^2)^{\frac{1}{2}}]\, e^{-i\sigma z} d\sigma.$$

The K_o integral appearing here may be evaluated by completing the contour around the boundaries of the fourth quadrant in the ω-plane for $z > 0$ with result

$$Q_-(z) = -\frac{i}{\pi}\int_0^\infty K_o[-ir(\tau^2+\lambda^2)^{\frac{1}{2}}]\, e^{-\tau z} d\tau - i\int_0^\lambda J_o[r(\lambda^2-\sigma^2)^{\frac{1}{2}}]\, e^{-i\sigma z} d\sigma. \tag{A 3}$$

Upon adding (A 2) and (A 3) and using the relation $K_o(-ix) = K_o(ix) + \pi i J_o(x)$ for x real and positive, we may arrive at the result

$$2G(z) = Q_+ + Q_- = \int_0^\infty J_o(ru)\, e^{-z(u^2-\lambda^2)^{\frac{1}{2}}} \frac{u\,du}{(u^2-\lambda^2)^{\frac{1}{2}}},$$

where $\arg(u^2-\lambda^2) = \pi$ if $u < \lambda$. This last integral appears in Luke (1962, p. 326) and has the value

$$(r^2+z^2)^{-\frac{1}{2}} \exp[-i\lambda(r^2+z^2)^{\frac{1}{2}}].$$

Since the result is even in z, it holds for z negative as well as z positive.

Since $A(\sigma) = A_1(\sigma) + iA_2(\sigma) = 0$ for $|\sigma| > \lambda$, the complete result for (A 1) is obtained. Upon separation into real and imaginary parts this yields expressions (16) and (16*a*), respectively.

Next we examine the behaviour of

$$\int_0^b s(t) \frac{\cos\lambda[r^2+(z-t)^2]^{\frac{1}{2}}}{[r^2+(z-t)^2]^{\frac{1}{2}}} dt$$

as $r \to 0$, for $0 < z < b$.

This integral may be rewritten as

$$\int_0^b s(t) \frac{dt}{[r^2+(z-t)^2]^{\frac{1}{2}}} - \int_0^b s(t) \left\{\frac{1-\cos[\lambda(r^2+(z-t)^2)^{\frac{1}{2}}]}{[r^2+(z-t)^2]^{\frac{1}{2}}}\right\} dt.$$

The first term is in the form usually encountered in slender-body theory (where $s = S''(z)$ instead of $S''+\lambda^2 S$ as here), and may be approximated as in (32) as $r \to 0$. In particular, it is of the order $\log r$ as $r \to 0$. On the other hand, as $r \to 0$, $z \to t$, the kernel of the second integral is bounded by $2\lambda/\pi$, so the second integral is $O(1)$ as $r \to 0$, hence (17) results.

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